Chapter 3.

Basic Probability.

$\mathbb{P}\{E\cup F\} = \mathbb{P}\{E\} + \mathbb{P}\{F\} - \mathbb{P}\{E\cap F\}$

$\mathbb{P}\{E\mid F\} = \frac{\mathbb{P}\{E\cap F\}}{\mathbb{P}\{F\}}$

Independent $E\perp F \Rightarrow \mathbb{P}\{E\cap F\} = \mathbb{P}\{E\}\mathbb{P}\{F\}$

conditionally independent $\Rightarrow \mathbb{P}\{E\cap F\mid G\} = \mathbb{P}\{E\mid G\}\,\mathbb{P}\{F\mid G\}$

Law of Total Probability $\mathbb{P}\{E\} = \sum_{i=1}^{n} \mathbb{P}\{E\mid F_i\}\mathbb{P}\{F_i\}$ for F_i partition

Bayes Law $\mathbb{P}\{F\mid E\} = \frac{\mathbb{P}\{E\mid F\}\mathbb{P}\{F\}}{\mathbb{P}\{E\}}$

Discrete Random Variables.

pmf $p_X(a) = \mathbb{P}\{X=a\}$

cdf $F_X(a) = \sum_{x\le a} p_X(x) = \mathbb{P}\{X\le a\}$

$\bar{F}_X(a) = 1 - F_X(a)$

Continuous Random Variables

pdf $\mathbb{P}\{a\le X\le b\} = \int_a^b f_X(x)dx$

cdf $F_X(a) = \mathbb{P}\{-\infty < X \le a\} = \int_{-\infty}^{a} f_X(x)dx$

$\bar{F}(a) = 1 - F_X(a) = \mathbb{P}\{X > a\}$

$E[X^i] = \sum_x x^i p_X(x)$ discrete

$= \int_{-\infty}^{\infty} x^i f_X(x)\,dx$ continuous

$Var(X) = E[(X-E[X])^2] = E[X^2] - E[X]^2$

Distribution	pmf $p_X(x)$	$E[X]$	$Var(X)$
Bernoulli(p)	$p_X(0) = 1-p$, $p_X(1) = p$	p	$p(1-p)$
Binomial(n,p)	$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Geometric(p)	$p_X(x) = (1-p)^{x-1}p$	$1/p$	$\frac{1-p}{p^2}$
Poisson(λ)	$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ

Distribution	pdf $f_X(x)$	$E[X]$	$Var(X)$
Exponential(λ)	$f_X(x) = \lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Uniform(a,b)	$f_X(x) = \frac{1}{b-a}$ $a\le x\le b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Pareto(α), $0<\alpha<2$	$f_X(x) = \alpha x^{-\alpha-1}$ $x>1$	∞ $\alpha\le 1$; $\frac{\alpha}{\alpha-1}$ $\alpha>1$	∞
Normal(μ,σ^2)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

$p_{X,Y}(x,y) = \mathbb{P}\{X=x \,\&\, Y=y\}$

$\int_c^d\int_a^b f_{X,Y}(x,y)\,dx\,dy = \mathbb{P}\{a<X<b \,\&\, c<Y<d\}$

$p_{X\mid A}(x) = \mathbb{P}\{X=x\mid A\} = \frac{\mathbb{P}\{X=x\cap A\}}{\mathbb{P}\{A\}}$

$f_{X\mid A}(x) = \begin{cases}\frac{f_X(x)}{\mathbb{P}\{X\in A\}} & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$

$X\perp Y \Rightarrow E[XY] = E[X]E[Y]$

$X\perp Y \Rightarrow Var(X+Y) = Var(X) + Var(Y)$

$E[X+Y] = E[X] + E[Y]$

$X\sim Normal(\mu,\sigma^2)$, $Y = aX+b \Rightarrow Y\sim Normal(a\mu+b, a^2\sigma^2)$

X_i iid w/ mean μ, variance σ^2, then $Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}}$

has $\forall z$, $\lim_{n\to\infty}\mathbb{P}\{Z_n\le z\} = \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}dx$

X_i iid, $S = \sum_{i=1}^{N} X_i$ and $N\perp X_i$, then

$E[S] = E[N]E[X]$

$E[S^2] = E[N]\,Var(X) + E[N^2]\,E[X]^2$

$Var(S) = E[N]\,Var(X) + Var(N)\,E[X]^2$

Chapter 4.

Inverse-Transform $u\in U(0,1)$, $X = F_X^{-1}(u)$

Accept-Reject (discrete)

find Q, $q_j>0$ iff $p_j>0$

$j\leftarrow$ instance of Q $\qquad c = \max \frac{p_j}{q_j}$

generate $U(0,1)$

if $U < \frac{p_j}{cq_j}$, return $P=j$

Accept-Reject (continuous)

find Y, $f_Y(t)>0$ iff $f_X(t)>0$

$t\leftarrow$ instance of Y $\qquad c = \max\frac{f_X(t)}{f_Y(t)}$

return $X=t$ with probability $\frac{f_X(t)}{cf_Y(t)}$

Chapter 5.

X_i iid, $S_n = \sum_{i=1}^n X_i$, $Y_n = \frac{S_n}{n} \Rightarrow Y_n \xrightarrow{P} E[X]$ as $n\to\infty$ } Weak Law of Large Numbers

i.e. $\forall k>0$ $\lim_{n\to\infty}\mathbb{P}\{|Y_n - E[X]|>k\} = 0$

X_i iid, $S_n = \sum_{i=1}^n X_i$, $Y_n = \frac{S_n}{n} \Rightarrow Y_n \xrightarrow{as} E[X]$ as $n\to\infty$ } Strong Law of Large Numbers

i.e. $\forall k>0$ $\mathbb{P}\{\lim_{n\to\infty}|Y_n - E[X]|\ge k\} = 0$

$\bar{N}^{Time} = \lim_{t\to\infty}\frac{\int_0^t N(v)dv}{t}$

$\bar{N}^{Ensemble} = \lim_{t\to\infty} E[N(t)] = \sum_{i=0}^{\infty} i p_i$, $p_i = \lim_{t\to\infty}\mathbb{P}\{N(t)=i\}$

For an ergodic system, $\bar{N}^{Ensemble}$ exists, and with probability 1, $\bar{N}^{Time} = \bar{N}^{Ensemble}$

Ergodic = positive recurrent, aperiodic, irreducible

Chapter 6.

Little's Law ergodic open system $\Rightarrow E[N] = \lambda E[T]$, $\bar{N}^{Time} = \lambda \bar{T}^{Time}$

Little's Law ergodic closed system $\Rightarrow N = X\,E[T]$, $E[R] = \frac{N}{X} - E[Z]$

Forced Flow Law $X_i = E[V_i]\cdot X$

Bottleneck Law $\rho_i = X\,E[D_i]$

Utilization Law $\rho_i = \frac{\lambda_i}{\mu_i} = \lambda_i E[S_i]$

Chapter 7.

Closed, interactive with N terminals,

$X \le \min\left(\frac{N}{D+E[Z]}, \frac{1}{D_{max}}\right)$

$E[R] \ge \max(D, N\cdot D_{max} - E[Z])$

first term for small N,

second term for large N

Chapter 8.

For DTMC,

$\mathbb{P}\{X_{n+1}=j\mid X_n=i_n, X_{n-1}=i_{n-1},\dots\} = \mathbb{P}\{X_{n+1}=j\mid X_n=i_n\}$ (Markovian)

$= P_{ij}$ (stationarity)

$\pi_j = \lim_{n\to\infty}P_{ij}^n$ = limiting probability that the chain is in state j independent of starting state i

$\vec{\pi} = (\pi_0, \pi_1, \dots, \pi_{M-1})$ = limiting distribution

Stationary Equations: $\vec{\pi}P = \vec{\pi}$, $\sum_{i=0}^{M-1}\pi_i = 1$

If $\vec{\pi}$ exists, it is also a stationary distribution

Chapter 9.

For state j, period$(j) = \gcd\{n \mid P_{jj}^n > 0\}$

called aperiodic if period $=1$

chain periodic if all states periodic

$i\to j$ if $\exists n$ $P_{ij}^n>0$, j accessible from i

if $i\leftrightarrow j$, then i and j communicate

chain irreducible if all states communicate

m_{ij} = mean # steps to first get to j from i

irreducible, aperiodic, finite state $\Rightarrow m_{jj} = \frac{1}{\pi_j}$

$f_j = \mathbb{P}\{$chain starting in j ever returns to $j\}$

$f_j = 1 \Rightarrow j$ recurrent $\Leftarrow \sum_{n=0}^{\infty}P_{ii}^n = \infty$

$f_j < 1 \Rightarrow j$ transient $\Leftarrow \sum_{n=0}^{\infty}P_{ii}^n < \infty$

i recurrent, $i\leftrightarrow j$ $\Rightarrow j$ recurrent } $\Rightarrow$ irreducible

i transient, $i\leftrightarrow j$ $\Rightarrow j$ transient } $\Rightarrow$ all same

positive recurrent $\Rightarrow m_{jj}$ finite

null recurrent $\Rightarrow m_{jj}$ infinite

Summary: irreducible, aperiodic

$\Rightarrow$ either all transient or all null recurrent, $\pi_j = 0\ \forall j$, $\nexists$ stationary

or $\pi_j = \frac{1}{m_{jj}}$, $\exists$ stationary, all positive recurrent

positive recurrent, irreducible $\Rightarrow$ with probability 1, $p_j = \frac{1}{m_{jj}} = \pi_j$ only if aperiodic too

renewal process: times between events iid $\sim F$

Renewal Thm. for renewal process with mean time between renewal $E[X]$

$\lim_{t\to\infty}\frac{N(t)}{t} = \frac{1}{E[X]}$ with probability 1

Time reversible: $\sum\pi_i = 1$ and $\pi_i P_{ij} = \pi_j P_{ji}$

irreducible $\Rightarrow$ all states same period

irreducible, period $d<\infty$, $\exists\vec{\pi}$ $\Rightarrow$ chain positive recurrent

Chapter 11.

$f(x) = \lambda e^{-\lambda x}$ $x\ge 0$, 0 otherwise

$X\sim Exp(\lambda) \Rightarrow F(x) = 1 - e^{-\lambda x}$ $x\ge 0$, 0 otherwise

$\bar{F}(x) = e^{-\lambda x}$ $x\ge 0$

$r(t) = \frac{f(t)}{\bar{F}(t)}$

$X_1\sim Exp(\lambda_1)$, $X_2\sim Exp(\lambda_2)$, $X_1\perp X_2 \Rightarrow \mathbb{P}\{X_1<X_2\} = \frac{\lambda_1}{\lambda_1+\lambda_2}$

$X_1\sim Exp(\lambda_1)$, $X_2\sim Exp(\lambda_2)$, $X = \min(X_1,X_2) \Rightarrow X\sim Exp(\lambda_1+\lambda_2)$

Poisson Process with rate $\lambda = PP(\lambda) =$

def 1 { ① $N(0) = 0$
② independent increments $N(t_i) - N(t_j) \perp N(t_k) - N(t_l)$
③ stationary increments, which is implied by $\forall s,t\ge 0$, $\mathbb{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

def 2 { seq of events such that interarrival times are iid $\sim Exp(\lambda)$, $N(0)=0$

def 3 { ① $N(0) = 0$
② stationary, independent increments
③ $\mathbb{P}\{N(\delta)=1\} = \lambda\delta + o(\delta)$
④ $\mathbb{P}\{N(\delta)\ge 2\} = o(\delta)$

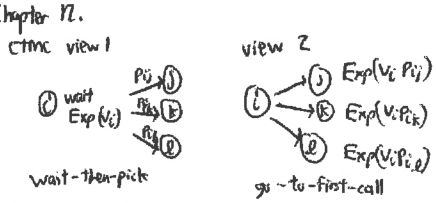
$PP(\lambda_1)$ merge $PP(\lambda_2) \sim PP(\lambda_1+\lambda_2)$

$PP(\lambda)$ type A with probability p, type B with probability $1-p$ $\Rightarrow$ type A $PP(p\lambda)$, type B $PP((1-p)\lambda)$

Uniformity Given one event of PP occurs by t, equally likely anywhere $[0,t]$

Chapter 12.

CTMC



Consider δ-flips

Chapter 13.

In M/M/1, $E[N] = \frac{\rho}{1-\rho}$ $\quad Var(N) = \frac{\rho}{(1-\rho)^2}$

$E[T] = \frac{1}{\mu-\lambda}$ $\quad E[T_Q] = \frac{\rho}{\mu-\lambda}$

PASTA = Poisson Arrivals See Time Averages, i.e. $a_n = p_n$

Chapter 14.

In M/M/k/k, $P_{block} = \frac{\mathbb{P}\{X=k\}}{\mathbb{P}\{X\le k\}}$ for $X\sim Poisson\left(\frac{\lambda}{\mu}\right)$

system utilization $\rho = \frac{\lambda}{k\mu}$

resource requirement $R = \frac{\lambda}{\mu}$

$P_{block} = \frac{(1-\rho)P_Q}{1-\rho P_Q}$

Chapter 15.

If M/M/k, arrival rate λ, server speed μ, $R = \frac{\lambda}{\mu}$ large,

k_α^* least number of servers to ensure $P_Q^{M/M/k} < \alpha$

then $k_\alpha^* \approx R + c\sqrt{R}$, where c solves $\frac{c\,\Phi(c)}{\phi(c)} = \frac{1-\alpha}{\alpha}$ (Φ: normal cdf, ϕ: normal pdf)